

A Rational Expectations Solution Method for the TRYM model

Craig Louis

Commonwealth Treasury

Abstract: This paper describes a rational expectations solution algorithm that is both efficient and relatively quick to construct. The method has been developed to solve the Treasury Macroeconomic (TRYM) model of the Australian economy under the assumption that agents in the financial sector have rational expectations. A Gauss-Newton algorithm suggests values for expectations that minimise the sum of squared expectational errors allowing the rest of the model to be solved in a recursive fashion. The main advantage of this particular methodology is that requires relatively little work to construct the numerical Jacobean matrix used by the algorithm.

INTRODUCTION

The TRYM model¹ is a medium sized macro economic model of the Australian economy that has recursive (adaptive) expectations specified in the household and business sectors and “quasi-rational” expectations specified in the financial sectors. That is, financial sector agents form their expectations of the future price level and exchange rate (both ten years ahead) in a forward looking fashion. However rather than setting these expectations to be consistent with the dynamic path taken by the model, they are set at values taken from a corresponding steady state simulation.

A slight inconsistency becomes apparent when one realises that the specification of the term structure of interest rates (between ten year bonds and ninety day bank bills) has been done in a backward looking fashion. The existing specification does not ensure the equivalence in returns from investing in a sequence of ninety day bank bills for ten years with holding a ten year bond for its duration. This situation can produce unbelievable results from some shocks and forecasting applications, simply because it can lead to large and persistent arbitrage opportunities across the yield curve in the model.

It is obviously desirable to have rational expectations specified in the financial markets in particular, as this sector faces the largest monetary incentives to form accurate expectations. While the present quasi-rational specification produces accurate approximations of fully rational results under some circumstances (monetary policy shocks for example), at the other extreme can produce totally unrealistic results in other situations (a permanent change in government outlays for example). The model has been run with rational expectations using a shooting procedure in the past (see for example Ryder et. al. (1993)), but this was very time consuming and computationally cumbersome.

Given the desire to have rational expectations specified in the model as a standard option, the challenge was to find a solution method that was efficient as well as convenient to maintain. This has been achieved by following a modified version of the extended path methodology suggested by Fair and Taylor (1983) and employing a Newtonian style algorithm that uses a numerical Jacobean matrix. The practicality of this approach has been enhanced by reducing the number of observations over which the entire model must be simulated by ten years and by making some simplifying assumptions when constructing the numerical Jacobean.

When simulating the model with rational expectations the forward looking expectational variables are treated as being exogenous for each iteration of the algorithm. Therefore, a method of solving the model in a recursive fashion, that is observation by observation moving forwards through time is required. For this purpose the TRYM model is solved using the package Time Series Processor (TSP) version 4.3. Such a package also needs to be able to conduct basic matrix operations such as multiplication and inversion.

In the following section of the paper the algorithm will be briefly described, the next section will outline how the Jacobean matrix has been constructed and discuss the approximations used to reduce the number of model simulations needed to construct it. The next section describes how the terminal conditions have been treated and outlines a suggestion that reduces the number of time periods over which the model must be simulated.

¹ See The Macroeconomics of the TRYM Model (1996).

THE ALGORITHM

The model to be solved can be represented as follows;

$$y_t = f \{y_{t-1}, y_{t-2}, \dots, y_{t-h}, x_t, x_{t-1}, \dots, x_{t-m}, E_t(z_{t+s})\} \quad (1)$$

where, y_t is the vector of endogenous variables at time t,
 x_t is the vector of exogenous variables at time t,
 z_{t+a} is a subset of the endogenous variables (the expectational variables), and
 $E_t(z_{t+s})$ is the expected value of these expectational variables at time t+s, formed at time t.

In the financial sector of the TRYM model the three expectational variables (z_{t+s}) are the ten year ahead price level (PGNEA) and exchange rate (RETWI) and the ten year geometric average of the 90 day bank bill interest rate. If the average 90 day interest rate is thought of as being calculated ex poste, then all three expectational variables can be treated as looking ten years ahead of the time they are formed.

The specification of the TRYM model (with quasi-rational expectations in the financial sector) can be summarised as being eighty one identities, and twenty nine behavioural equations which are almost entirely specified as error correction models (ECMs) or partial adjustment models. Overall the model is almost linear in terms of its explanatory variables. The slight nonlinearities that are present come from the wages equation which is specified as an expectations augmented Phillips curve, and the unfilled vacancies, unemployment equation which influences employment and average hours worked.

The objective function to be minimised is the sum of squared expectational errors, or

$$\sum_{t=s}^n (z_t - E_{t-s}(z_t))^2 \quad (2)$$

where s is the number of time periods ahead that the expectations are looking (in TRYM s =ten years or 40 quarters).

Starting with three initial “guesses” of the paths taken by the expectational variables, the model is simulated treating these expectations as though they were exogenous. Using the results from this simulation, the expectational errors are calculated and stacked into a vector; as follows,

$$\boldsymbol{\varepsilon}^1 = \begin{bmatrix} z_t - E_{1-s}(z_1) \\ z_2 - E_{2-s}(z_2) \\ \cdot \\ \cdot \\ z_n - E_{n-s}(z_n) \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} \boldsymbol{\varepsilon}^1 \\ \boldsymbol{\varepsilon}^2 \\ \boldsymbol{\varepsilon}^3 \end{bmatrix} \quad (3)$$

where the superscript on \mathcal{E} denotes which of the three expectational variables the path of errors corresponds to (either the exchange rate, the price level or interest rates). The order in which the variables are stacked is irrelevant as long as it is consistent with that used when constructing the Jacobean matrix (described below).

A set of suggested changes to the paths taken by the expectational variables through the simulation period is then given by the familiar ordinary least squares regression expression;

$$\begin{bmatrix} \Delta E(z^1) \\ \Delta E(z^2) \\ \Delta E(z^3) \end{bmatrix} = (X' X)^{-1} X' E \quad (4)$$

where X is a Jacobean matrix (the construction of which is explained in the next section), and each of the three $\Delta E(Y^i)$ s represents a vector of alterations to the path taken for the corresponding (ith) expectational variable. For example;

$$\Delta E(z^1) = \begin{bmatrix} \Delta E_{1-s}(z_1) \\ \Delta E_{2-s}(z_2) \\ \cdot \\ \cdot \\ \Delta E_{n-s}(z_n) \end{bmatrix}. \quad (5)$$

Having obtained the set of suggested changes to the original values for the expectational variables, the first iteration of the algorithm is complete and the model can be simulated once again treating these variables as exogenous. This process is repeated until the value of the objective function has been reduced to a sufficiently small amount.

CONSTRUCTION OF THE JACOBEAN MATRIX

The Jacobean is constructed in a fashion that minimises the number of simulations of the model required before the algorithm can be run. The Jacobean must be rebuilt every time the model's parameters are updated. Once this has been done it appears that the one matrix is sufficient to solve the rational expectations problem for any reasonable size shock, (again following from the absence of significantly large non-linearities in the model).

The first step in this process is to run a simulation of the model using any broadly sensible profiles for the expectational variables. For this purpose the original quasi-rational specification in TRYM was a good starting point. The simulation should be conducted over a sample period that is $n+s$ periods long, where n is at least twice the size of s . In this example where the expectations are formed for values ten years ahead, the simulations are run over a thirty year period.

The second step is to calculate the error vectors \mathcal{E} for each of the expectational errors, (as in equation (3) above).

$$(z_t - E_{t-s}(z_t))_{baseline} \quad (6)$$

Next, three more simulations need to be run where each of the expectational variables are altered in the first quarter by a small perturbation (by 0.01 for example). The corresponding error vectors are calculated for each of the expectational variables and numerical differentials are formed for each of the expectational variables;

$$\frac{d\mathcal{E}_t^i}{dE_1^j} = \frac{d(z_t - E_{t-s}(z_t))}{dE_1(z_s)} = \frac{[(z_t - E_{t-s}(z_t))_{baseline} - (z_t - E_{t-s}(z_t))_{shock}]}{0.01} \quad (7)$$

where $t=1,2,\dots,n+s$
 $i=1,2,3$
 $j=1,2,3$.

It is important to note that the calculation of the derivatives includes the effect of changing one expectational variable on the errors associated with the other expectational variables. This means that in the TRYM example a total of nine numerical deviations are calculated from the four simulations conducted.

By making the assumption that the differential of the expectational errors is the same through time regardless of the state of the model eliminates the need to run any more simulations of the model when constructing the Jacobean matrix. This is equivalent to assuming that the model is linear in terms of its explanatory variables and represents a slight approximation of the true nature of the TRYM model. More formally, this assumption can be written as;

$$\frac{d\mathcal{E}_t^i}{dE_1^j} = \frac{d\mathcal{E}_{t+1}^i}{dE_2^j} = \frac{d\mathcal{E}_{t+2}^i}{dE_3^j} = \dots = \frac{d\mathcal{E}_{t+s}^i}{dE_n^j} \quad (8)$$

So far we have simulated and assumed our way to deriving a derivative vector of each expectational error with respect to a variation in each of the expectational variables, at every observation in the simulation period. All that remains is to construct the Jacobean matrix in a fashion that corresponds with the error vector, as described in equation (3). This is probably best described by breaking the matrix up into nine sub-matrices, each representing a smaller Jacobean matrix of each expectational error for a small change in either of the three expectational variables. For example the first sub-matrix would represent the change in the expectational error for the first expectational variable with respect to a change in that variable, ie;

$$X^{11} = \begin{bmatrix} \frac{d\mathcal{E}_1^1}{dE_1^1} & \frac{d\mathcal{E}_1^1}{dE_2^1} & \frac{d\mathcal{E}_1^1}{dE_3^1} & \cdot & \frac{d\mathcal{E}_1^1}{dE_n^1} \\ \frac{d\mathcal{E}_2^1}{dE_1^1} & \frac{d\mathcal{E}_2^1}{dE_2^1} & \cdot & \cdot & \frac{d\mathcal{E}_2^1}{dE_n^1} \\ \frac{d\mathcal{E}_3^1}{dE_1^1} & \frac{d\mathcal{E}_3^1}{dE_2^1} & \cdot & \cdot & \frac{d\mathcal{E}_3^1}{dE_n^1} \\ \frac{d\mathcal{E}_n^1}{dE_1^1} & \frac{d\mathcal{E}_n^1}{dE_2^1} & \cdot & \cdot & \frac{d\mathcal{E}_n^1}{dE_n^1} \end{bmatrix} \quad (9)$$

So that the first column of this submatrix represents the effect of changing the first expectational variable (E_1^1) in the first observation of the sample on the expectational error from that variable (\mathcal{E}_t^1) through the sample period.

To use an example from the TRYM context, the first number in the first column may be the impact of altering the expected future exchange rate in the first quarter, on the exchange rates' expectational error in the first quarter. The second observation in this column would then be the subsequent effect on this expectational error in the second quarter occurring from the same variation in the expected exchange rate in the first quarter.

The second column in the above example represents the sequence of effects on the expectational errors for the exchange rate through out the simulation period of altering the expected future exchange rate in the second quarter.

The next submatrix to the right represents the effect of a variation in the second expectational variable on the errors arising from the first. Continuing the above example, each column of this matrix would represent the effect on the exchange rates expectational errors of changing the expected future price level in each particular quarter.

$$X^{12} = \begin{bmatrix} \frac{d\varepsilon_1^1}{dE_1^2} & \frac{d\varepsilon_1^1}{dE_2^2} & \frac{d\varepsilon_1^1}{dE_3^2} & \frac{d\varepsilon_1^1}{dE_n^2} \\ \frac{d\varepsilon_2^1}{dE_1^2} & \frac{d\varepsilon_2^1}{dE_2^2} & \cdot & \frac{d\varepsilon_2^1}{dE_n^2} \\ \frac{d\varepsilon_3^1}{dE_1^2} & \frac{d\varepsilon_3^1}{dE_2^2} & \cdot & \frac{d\varepsilon_3^1}{dE_n^2} \\ \frac{d\varepsilon_n^1}{dE_1^2} & \frac{d\varepsilon_n^1}{dE_2^2} & \cdot & \frac{d\varepsilon_n^1}{dE_n^2} \end{bmatrix} \quad (10)$$

The overall Jacobean matrix can now be constructed by placing each of the nine sub-matrices in the positions that correspond to the elements in the error vector, as represented in equations (3) and (4).

$$X = \begin{bmatrix} X^{11} & X^{12} & X^{13} \\ X^{21} & X^{22} & X^{23} \\ X^{31} & X^{32} & X^{33} \end{bmatrix} \quad (11)$$

This matrix will be of dimension $(n \times 3) \times (n \times 3)$, which in the case of the TRYM example where the simulation period has been chosen to be twenty years long at a quarterly frequency, this matrix is quite large (240×240) . The choice of sample period is crucially dependant on the distance forward through time that the expectations are looking and the treatment of the models terminal conditions.

THE TERMINAL CONDITIONS

The reader who is either very observant or familiar with rational expectations solution problems may have noticed the omission, so far, of any mention of the terminal conditions. The treatment of the terminal conditions is an interesting facet of any rational expectations solution method as it represents a classic "chicken or the egg" problem that must be overcome for the solution method to be operational.

The problem arises because the modeller is trying to eliminate the difference between the expected future value of a variable and the actual value that the variable takes right throughout a finite

simulation period. The problem is : What happens when the simulation period is almost over and the expectational variables are looking past the end of the simulation?

The treatment of the terminal conditions in the Fair Taylor Extended Path methodology is to extend the simulation path beyond the end of the desired (or original) simulation period by at least twice the length of time that the expectational variables look ahead. At the end of this simulation period it is usual practice to assume that the expectational variables and the corresponding expectations of these variables jump to their steady state values. This period at the end of the simulation period can be thought of as representing the rest of eternity as far as the simulation is concerned and must be far enough beyond the first part of the simulation to not effect the results in the original desired sample period.

In the TRYM context, this approach would imply simulating the model repetitively over a sample period that is at least thirty years long and would expand the Jacobean matrix's dimensions to greater than (360×360) , which is beyond the limits of the TSP software used. Thus following the Extended Path approach would severely reduce the practicality of employing rational expectations in the TRYM model.

The solution adopted here is to extend the sample period by the length of time that the expectational variables look forward, ten years, and replace the following ten years worth of values for the expectational variables and their associated expectations with that suggested by reduced form expressions that replicates the model's normal correction back to it's steady state values after a shock. Initial trials using error correction models explaining the three expectational variables in terms of lagged dependant variables and their steady state values have proved to be quite proficient at replicating the model's dynamics.

The three reduce form expressions for the values taken by the exchange rate, the 90 day bank bill yield and the GNEA deflator past the end of the twenty year simulation period are as follows;

$$ER_t = ER_{t-1} + 0.9 \times (ER_{t-1} - ER_{t-2}) + 0.015 \times \{ERSS_{t-1} - ER_{t-1}\} \quad (12)$$

$$RI90_t = RI90_{t-1} + 0.97 \times (RI90_{t-1} - RI90_{t-2}) + 0.03 \times \{RI90SS_{t-1} - RI90_{t-1}\} \quad (13)$$

$$P_t = P_{t-1} \times \exp(\text{inf}) + 0.89 \times (P_{t-1} - P_{t-2} \times \exp(\text{inf})) + 0.016 \times \{PSS_{t-1} - P_{t-1}\} \quad (14)$$

where ER is the exchange rate
 RI90 is the ninety day bank bill
 P is the GNEA deflator
 inf is the steady state inflation rate
 the tag of SS represents a steady state value for that variable

These estimated parameters are from a version of the TRYM model estimated using the September quarter 1995 TRYM model data base and reflect the typically slow adjustment of the TRYM model towards its steady state growth path.

In the first run of the model with rational expectations values for the reduced form parameters reported above can be estimated from a standard deterministic simulation of the model using the quasi-rational expectations (alternatively any semi sensible numbers can be put in their place). It is then simply a matter of updating these estimated parameters using the first ten years from the first

rational expectations solution. If desired, these estimates can be further refined by repeating this process until the parameter values cease to change.

Different lag structures may be required to accurately replicate a different models convergence to it's steady state growth path. The quality of this replication should be tested by dynamically simulating the reduced form equations and comparing the results with those from the full model..

CONCLUSION

Given the desire to specify rational expectations in the TRYM model and the constraints of computing power and/or a lack of computer programming skills, a practical method of solving the model with rational expectations has been suggested.

This approach greatly reduces the number of simulations needed to construct the numerical Jacobean matrix used by the algorithm. This matrix only needs to be rebuilt when the models parameters or specification are altered. The method also reduces the size of the simulation period required to insulate the sample period of interest from the effects of eventually jumping to the models terminal conditions.

Whilst the computational costs of constructing and running the algorithm have been reduced to a minimum, initial trials on the TRYM model using several different parameter and data sets suggest the convergence properties of the algorithm are quite respectable, with convergence typically being achieved within four to ten iterations.

REFERENCES

Downes et.al.,The Macroeconomics of the TRYM Model, Commonwealth Treasury 1996.

Fair Ray C. and Taylor John B., Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models, *Econometrica* Vol 51, No.4, July 1983.

Harvey A., The Econometric Analysis of Time Series, Philip Allan 1990.

Murphy C.W., A Quick Solution Method for Non-Linear Rational Expectations Models, memo Australian National University (date unknown approx 1990)

Ryder B, Johnson A, Taplin B, Jilek P., Australia's Trade Linkages with the World, TRYM Conference Paper Number 6. (1993)